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The generation of Langmuir circulations by an instability mechanism

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Equations governing the current system in the upper layers of oceans and lakes were derived by Craik & Leibovich (1976). These incorporate the dominant effects of both wind and waves. Solutions comprising the mean wind-driven current and a system of 'Langmuir' cells aligned parallel to the wind were found for cases in which the wave field consisted of just a pair of plane waves. However, it was not clear that such cellular motions would persist for the more realistic case of a continuous wave spectrum.

The present paper shows that, in the latter case, infinitesimal spanwise periodic perturbations will grow on account of an instability mechanism. Mathematically, the instability is closely similar to the onset of thermal convection in horizontal fluid layers. Physically, the mechanism is governed by kinematical processes involving the mean (Eulerian) wind-driven current and the (Lagrangian) Stokes drift associated with the waves. The relationship of this mechanism to instability models of Garrett and Gammelsrød is clarified.

1. Introduction

The importance of Langmuir circulations as a mixing process in the top few metres of lakes and oceans is being increasingly recognized. While there is little doubt that these circulations originate from an interaction of the mean wind-driven current and the wave field, the precise nature of this interaction is still the subject of debate. Indeed, there is a superfluity of theories purporting to explain the phenomenon, and these have been comprehensively reviewed by Craik (1970), Faller (1971) and Pollard (1976). At present, three recent but conflicting theories appear to command most support: those of Gammelsrød (1975), Craik & Leibovich (1976) and Garrett (1976).

Gammelsrød attributes the circulations to an instability of the wind-driven shear flow and his theory, like an earlier one of Faller (1964, 1966), depends on the influence of Coriolis acceleration: without the Coriolis term, no instability occurs. However, it would be surprising if the existence of Langmuir circulations with a typical length scale of several metres turned out to depend crucially on the Coriolis acceleration.

Garrett (1976) also proposes an instability mechanism, this time involving variations in wave amplitude and correspondingly distributed dissipation of wave energy. But his model is a very crude heuristic one. In particular, certain of his approximations appear to be inconsistent and terms neglected without firm justification may be comparable with those retained. Although the underlying 'physics' of Garrett's mechanism appear plausible, several aspects of his analysis are unacceptable (see the remarks of Leibovich & Radhakrishnan 1977).

Craik & Leibovich (1976), following earlier work of Craik (1970) and Leibovich & Ulrich (1972), provide a rational derivation of the equations likely to govern Langmuir circulations and the mean wind-driven current. They then consider the solution of a model problem in which the gravity-wave field is replaced by a single pair of wave trains propagating at oblique angles to the wind direction. A more complete numerical solution to this problem, in which all nonlinear terms of the equations are retained, has recently been obtained by Leibovich (1977) and Leibovich & Radhakrishnan (1977). The former paper concentrates on the development of the mean current from a zero initial state while the latter considers the structure of the growing 'Langmuir cells'. Various known features of actual Langmuir circulations are reproduced in these solutions.

In their model problem, periodic spanwise (cross-wind) variations of the Stokes drift associated with the assumed wave field drive correspondingly spaced 'Langmuir cells' by distorting the originally spanwise vorticity imparted to the water by a constant wind stress. Similar forced cellular motions will arise whenever the wave field is such as to cause significant spanwise variations in the Stokes drift. However, a difficulty arises when the wave field comprises a continuous spectrum of wavenumbers rather than an assemblage of discrete components; for, if the Fourier components of the spectrum have random phases, the spanwise variations of the Stokes drift must phase mix to zero. While it is not clear that, in practice, the phases must be random, one cannot make progress with a statistical model without this assumption. Such a statistical model developed by Craik & Leibovich (1976) yields a plausible estimate of likely cell spacing; but it predicts infinitesimally weak secondary currents as the number of wave components increases to infinity (i.e. as a continuous spectrum is approached). In short, the model problem solved by Craik, Leibovich and Radhakrishnan successfully reproduces known features of Langmuir circulations and, more important, elucidates the inherent dynamical processes; but its *direct* relevance to the ocean is open to question. On the other hand, this model may well be appropriate for confined bodies of water such as small lakes where the fetch- or time-limited wave spectrum can be sharply peaked.

The present work arose from an attempt to reconcile the theories of Craik & Leibovich, Garrett and Gammelsrød. It was recognized that a basic process in both the Craik–Leibovich and the Garrett theories is the distortion of vorticity by the Stokes drift. But the basic difference remains that the Craik–Leibovich–Radhakrishnan solutions are forced motions in response to spanwise variations imposed by the wave field, whereas Garrett and Gammelsrød propose instability mechanisms in which spanwise variations arise spontaneously by amplification of initially infinitesimal disturbances.

Starting from Craik & Leibovich's governing equations, one may prescribe a Stokes drift which depends only on depth and pose a stability problem for spanwise-periodic motions. Unlike Garrett's and Gammelsrød's models, there is no need to include either distributed wave dissipation or the Coriolis force at this level of approximation. Instead, the motion is governed *only* by kinematic processes and by ordinary viscous or 'eddy-viscous' diffusion. The stability analysis has much in common with corresponding work in thermal convection and rotating fluids and these analogies will be described. The present physical model is among the simplest yet proposed and it yields an instability which at last may satisfactorily explain the regular spacing of Langmuir cells in the presence of a continuous spectrum of wind waves.

2. The governing equations

The equations of motion are precisely those derived by Craik & Leibovich (1976) and Leibovich (1977). These were obtained by rational approximation based on clearly stated hypotheses concerning the relative magnitudes of certain key quantities. Foremost among these is the hypothesis that steady currents are typically small compared with wave orbital speeds: a fact well established by observation. The reader is referred to these papers for full details of the derivation and approximations.

We here adopt the scaling of Leibovich (1977) in writing the velocity vector for the non-oscillatory water motions as

$$\frac{u_*^2}{\nu_T \kappa} u\mathbf{i} + \frac{u_* a(\omega \nu_T)^{\frac{1}{2}}}{\nu_T}(v\mathbf{j} + w\mathbf{k}),$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the windward (x) direction, spanwise (y) direction and upward (z) direction, the water being taken to fill the region $-\infty < z \leqslant 0$. Here u_* is the water friction velocity, which is related to a constant wind stress τ_w by $\tau_w = \rho u_*^2$, where ρ is the water density; ν_T is an eddy viscosity assumed to be constant; κ^{-1} is a length scale characteristic of the waves (κ is the wavenumber for the discrete wave model); a is a characteristic wave amplitude and ω a wave frequency scale which may be taken as $(g\kappa)^{\frac{1}{2}}$ without loss of generality, where g is the gravitational acceleration. The space variables x, y and z are non-dimensionalized relative to the length scale κ^{-1} and dimensionless time t is scaled by $(a\kappa u_*)^{-1}(\nu_T/\omega)^{\frac{1}{2}}$. The dimensionless velocity components u, v and w are functions of y, z and t but not of x since the current system is assumed to be uniform in the windward direction. These rather cumbersome scalings have the advantage of reducing the nonlinear equations and boundary conditions to their simplest form.

These equations are

$$u_t + vu_y + wu_z = La\,\nabla^2 u, \tag{2.1a}$$

$$\Omega_t + v\Omega_y + w\Omega_z = La\,\nabla^2\Omega + F, \tag{2.1b}$$

$$\Omega = -\nabla^2\psi, \quad \nabla^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2, \tag{2.1c}$$

$$v = \psi_z, \quad w = -\psi_y, \quad La = \frac{\nu_T \kappa}{a u_*}\left(\frac{\nu_T}{\omega}\right)^{\frac{1}{2}}, \tag{2.1d}$$

where v and w may be represented by the stream function ψ in virtue of continuity and Ω is the longitudinal (x) vorticity. The influence of the waves is manifested in the term F, which is given by

$$F = \mathscr{U}_y u_z - \mathscr{U}_z u_y, \tag{2.2}$$

where $\mathscr{U}$ is the dimensionless Stokes drift. This is defined as the time average

$$\mathscr{U} = \overline{\nabla(\mathbf{u}^w . \mathbf{i}) . \int^t \mathbf{u}^w dt}, \tag{2.3}$$

where $a\omega\mathbf{u}^w$ is the irrotational dimensional velocity vector associated with the wave motion. In terms of vorticity dynamics, F represents the distortion of vorticity by the Stokes drift as discussed above. The parameter La, appropriately named the Langmuir number by Leibovich, is effectively an inverse Reynolds number; it is typically small in situations of interest.

If one postulates a pre-existing quasi-steady wave field with time-independent Stokes drift but no Eulerian currents (u, v, w) and poses the initial-value problem corresponding to application of a constant wind stress τ_w on the surface $z = 0$ for all $t \geq 0$ (thereby avoiding consideration of how the wave field is generated), the appropriate boundary conditions for a developing current system with *fixed spanwise periodicity* $L\kappa^{-1}$ are (cf. Leibovich)

$$\begin{gathered}\Psi(0,z,t) = \Omega(0,z,t) = u_y(0,z,t) = 0,\\ \Psi(L,z,t) = \Omega(L,z,t) = u_y(L,z,t) = 0,\\ \Psi(y,0,t) = \Omega(y,0,t) = 0, \quad u_z(y,0,t) = 1,\\ \Psi, u \to 0 \quad (z \to -\infty),\\ \Psi(y,z,0) = u(y,z,0) = 0.\end{gathered}$$

Note that La and L are the only two parameters of the problem, but that in addition the Stokes drift $\mathscr{U}$ will depend on the properties of the chosen wave field. The particular function chosen in the Craik–Leibovich–Radhakrishnan model is

$$\mathscr{U} = [4-(\pi/L)^2]^{\frac{1}{2}}\, e^{2z}\{1+[1-(\pi/2L)^2]\cos(\pi y/L)\} \quad (L > \tfrac{1}{2}\pi),$$

which corresponds to a pair of monochromatic gravity waves propagating at angles $\pm\sin^{-1}(\pi/2L)$ to the x direction. Since this gives rise to a 'forcing term' F with specified y periodicity, the spacing of the cells is predetermined.

The present work shows that such y variation of the Stokes drift is not a prerequisite for development of spanwise-periodic currents. Rather, such currents can arise by instability in cases where the Stokes drift $\mathscr{U}$ is a function of depth only. The Stokes drift of a continuous spectrum of waves symmetric about the wind direction is just such a function.

The analysis to follow is in several parts. Section 3 poses a quasi-steady linear stability problem based on the assumption that the Stokes drift $\mathscr{U}$ and the mean Eulerian wind-driven current $u = \bar{u}$ may be treated as functions of depth only: the stability of small spanwise-periodic disturbances is then examined and inviscid solutions for some particular cases are given. Section 4 deals with the viscous problem and attention is there drawn to analogous solutions of the well-known stability problems in which (i) a layer of fluid is heated from below (or cooled from above) and (ii) concentric rotating cylinders are in relative motion.

Since the limitations of quasi-steady stability analyses, when the primary state varies with time, are well recognized, a combined initial-value stability problem is considered in §5: this has much in common with work of Foster (1965, 1968) on the onset of thermal convection. The relevant conclusions of Foster's studies are described and interpreted in the present context. The final section examines the physics of the instability mechanism and explores the relationship of the present mechanism to those of Garrett and Gammelsrød. The probable form of circulations with magnitudes beyond the range of validity of our linear theory is also discussed.

3. The quasi-steady stability problem

In the absence of circulations, the mean wind-driven current $u = \bar{u}(z,t)$ must satisfy the equation

$$\bar{u}_t - La\,\bar{u}_{zz} = 0,$$

along with the boundary and initial conditions

$$\bar{u}_z = 1 \quad (z = 0,\quad t \geqslant 0), \quad \bar{u} \to 0 \quad (z \to -\infty, t \geqslant 0),$$
$$\bar{u} = 0 \quad (z < 0, t = 0).$$

This has the solution (cf. Leibovich)

$$\bar{u} = 2\left(\frac{La\,t}{\pi}\right)^{\frac{1}{2}} [\exp(-\eta^2) - \pi^{\frac{1}{2}}\eta\,\mathrm{erfc}(+\eta)], \quad \eta = \frac{-z}{2(La\,t)^{\frac{1}{2}}}. \tag{3.1}$$

If the time scale associated with the evolution of this profile is long compared with characteristic growth times of unstable disturbances then a quasi-steady stability analysis is permissible. Assuming meantime that this is so, we treat both $\bar{u}$ and $\mathscr{U}$ as functions of z only and write

$$u = \bar{u}(z) + \epsilon\hat{u}(y,z,t), \quad \psi = \epsilon\Psi(y,z,t),$$

where ϵ is small, to obtain the linearized $O(\epsilon)$ equations

$$\left.\begin{aligned}(\partial/\partial t - La\,\nabla^2)\,\hat{u} &= \Psi_y \bar{u}_z,\\ (\partial/\partial t - La\,\nabla^2)\,\nabla^2\Psi &= \hat{u}_y \mathscr{U}_z.\end{aligned}\right\} \tag{3.2}$$

A typical Fourier component is written as

$$(\bar{u}, \Psi) = \mathrm{Re}\,\{(U, -il^{-1}W)\exp(ily)\exp(\sigma t)\},$$

where l is its spanwise wavenumber and $\sigma(l)$ the growth or decay rate to be found. This yields

$$\left.\begin{aligned}[\sigma - La(D^2 - l^2)]\,U &= \bar{u}_z\,W,\\ [\sigma - La(D^2 - l^2)]\,(D^2 - l^2)\,W &= -l^2\mathscr{U}_z\,U,\end{aligned}\right\} \tag{3.3}$$

where $D \equiv d/dz$.

To demonstrate the existence of a possible instability mechanism, we first omit viscous terms by setting $La = 0$. Then

$$(D^2 - l^2)\,W = -(l^2\sigma^{-2}\bar{u}_z\mathscr{U}_z)\,W, \quad U = \bar{u}_z\sigma^{-1}W$$

and the inviscid boundary conditions are

$$W(0) = 0, \quad W \to 0 \quad (z \to -\infty).$$

Case I

One case yielding an immediate solution is

$$\bar{u}_z\mathscr{U}_z = \begin{cases}\Lambda & (-H \leqslant z \leqslant 0),\\ 0 & (-\infty < z < -H),\end{cases}$$

where Λ is a positive constant. This corresponds, for instance, to cases where $\bar{u}$ and $\mathscr{U}$ exhibit uniform shear to a depth H and then one or both functions are zero beyond that depth. We note that the profile (3.1) may be approximated by such a function. For $-\infty < z < -H$, W has the form $A\exp lz$ to satisfy the boundary condition at infinity. The matching conditions at $z = -H$ are that W and DW should be continuous while in $-H \leqslant z \leqslant 0$

$$D^2W = l^2(1 - \Lambda\sigma^{-2})\,W.$$

There is no non-trivial solution for $\Lambda < \sigma^2$. For $\Lambda > \sigma^2$ the solution in $-H \leqslant z \leqslant 0$ is

$$W(z) = -A \sin mz/e^{lH} \sin mH, \quad m^2 \equiv l^2(\Lambda\sigma^{-2} - 1),$$
$$mH \cot mH = -lH.$$

The last equation has an infinite number of positive roots $m = m_i$ $(i = 1, 2, \ldots)$ each corresponding to a permissible eigenstate. The smallest root $m = m_1$ for fixed lH is that associated with the largest growth rate

$$\sigma = \sigma_1(lH) = \Lambda^{\frac{1}{2}}(1m + {}_{1}^{2}/l^2)^{-\frac{1}{2}}.$$

The magnitude of σ_1 increases uniformly with l from zero when $l = 0$ to $\Lambda^{\frac{1}{2}}$ when $l \to \infty$: that is to say, *all* wavenumbers are unstable with the fastest growth rates apparently occurring at the smallest wavelengths. Of course, this conclusion will be modified by viscosity, which will assuredly stabilize the shortest wavelengths, but the existence of a mechanism of instability has been demonstrated.

Case II

A second class of profiles which yields a simple analytic solution is that where

$$\bar{u}_z \mathscr{U}_z = \Lambda e^{z/H}.$$

This class includes all cases where both $\mathscr{U}$ and $\bar{u}$ decrease exponentially (possibly at different rates) with depth and such profiles may be acceptable approximations for (3.1) and for the Stokes drift. The equation for W is now

$$D^2 W = l^2(1 - e^{z/H}\Lambda\sigma^{-2})\, W$$

and the change of variable

$$\zeta \equiv \Lambda^{\frac{1}{2}}(2lH/\sigma) \exp(z/2H)$$

leads directly to Bessel's equation

$$\zeta^2 W_{\zeta\zeta} + \zeta W_\zeta + W[\zeta^2 - (2lH)^2] = 0$$

with the boundary condition $W = 0$ at both $\zeta = 0$ and $\zeta = \Lambda^{\frac{1}{2}}(2lH/\sigma)$.

The solution is $W = AJ_\nu(\zeta)$, where $\nu = 2lH$, and the roots $\sigma = \sigma_i$ $(i = 1, 2, \ldots)$ are given by the condition

$$\sigma_i = \Lambda^{\frac{1}{2}}(2lH)/\zeta_i,$$

where the ζ_i are the zeros of $J_\nu(\zeta)$ with $\nu = 2lH$. The most unstable root for given lH is the root $\sigma = \sigma_1$ corresponding to the smallest positive zero ζ_1. As lH increases, σ_1 increases uniformly from zero at $lH = 0$ to $\Lambda^{\frac{1}{2}}$ as $lH \to \infty$ (see Abramowitz & Stegun 1965, chap. 9) just as in case I.

4. Viscous solutions, $La \neq 0$

On writing $\sigma = Lap$ in (3.3) we obtain

$$(D^2 - l^2 - p)\, U = -La^{-1}\bar{u}_z W,$$
$$(D^2 - l^2 - p)(D^2 - l^2)\, W = l^2 La^{-1} \mathscr{U}_z U,$$
$$W = D^2 W = DU = 0 \quad (z = 0), \quad W, U \to 0 \quad (z \to -\infty).$$

When $\bar{u}_z$ and $\mathscr{U}_z$ are positive *constants* and the depth of liquid is taken as *finite* $(-H \leqslant z \leqslant 0)$ rather than infinite with, say, a rigid boundary at $z = -H$, the problem is a variant of the classical Bénard stability problem for a heated layer of fluid with unit Prandtl number and a free upper surface. Now U plays the role of the temperature perturbation, the only departure from the classical problem being that the isothermal boundary condition $U(0) = 0$ is here replaced by $DU(0) = 0$. If we take $\bar{u}_z = 1$ and $\mathscr{U}_z = K$ and redefine the space variable as $z_1 = z/H$, the Rayleigh number Ra of the Bénard problem is identified with $La^{-2}KH^4$ in the present case.

A similar analogy exists with the Taylor instability problem for flow between concentric rotating cylinders in the 'small-gap' approximation. Further, the stability problem for flow between fixed concentric cylinders under a constant azimuthal pressure gradient is equivalent to the case $\bar{u}_z = (1+2z_1)$, $\mathscr{U}_z = -Kz_1(1+z_1)$. Details of all these related problems may be found in Chandrasekhar (1961).

On writing $H = \kappa d$, where d is the dimensional depth of fluid, and noting that $\mathscr{U}_z = K$, which must be $O(1)$, may be defined to be unity by readjusting the definition of κ, it is found that the 'effective Rayleigh number' is

$$Ra = La^{-2}H^4 = (a\kappa)^2\left(\frac{du_*}{\nu_T}\right)^2\left(\frac{d^2\omega}{\nu_T}\right). \tag{4.1}$$

Estimates believed to be typical of the ocean at a wind speed of about 10 m s^{-1} are (cf. Leibovich & Radhakrishnan's equation 17)

$$a\kappa \approx 0{\cdot}2, \quad \nu_T \approx 25\ \mathrm{cm^2\,s^{-1}}, \quad u_* \approx 1{\cdot}5\ \mathrm{cm\,s^{-1}}, \quad \omega \approx 1\ \mathrm{s^{-1}}, \tag{4.2}$$

which yield $Ra = 570d^4$, where d is measured in metres. Clearly, if the water depth is more than a few metres the critical Rayleigh number for onset of instability ($Ra = 1708$ for the Bénard problem with rigid boundaries, less than this with free boundaries) is sure to be exceeded. However, it must be borne in mind that this analogy requires constant gradients $\bar{u}_z$ and $\mathscr{U}_z$ whereas in practice these decay to zero with increasing depth.

Relevant to our initial-value problem is the work on the stability of time-dependent temperature profiles of Lick (1965) and Currie (1967), both of whom employ the quasi-steady approximation. In these studies, the mean temperature gradient is taken to be constant over part of the fluid layer and zero for the remainder, with constants chosen to model the heat-conduction solution at various times after heating is started. As might be expected, the (quasi-steady) critical Rayleigh number for onset, based on total depth, rapidly decreases from very large values as time increases and as the thermal boundary layer thickens; and instability occurs when sufficient time has elapsed to reduce this instantaneous critical value to that of the configuration, provided the latter exceeds about 1340. In effect, it is the thickness of the thermal boundary layer rather than the overall depth of the fluid layer which is the crucial length scale.

In accord with this conclusion, it seems reasonable in the present work to identify d not with the actual depth of water but with some $O(1)$ multiple of the *lesser* of (i) the length scale κ^{-1} over which most of the variations of $\mathscr{U}$ occur and (ii) the 'boundary-layer thickness' $2\kappa^{-1}(La\,t)^{\frac{1}{2}}$ associated with $\bar{u}$ in (3.1). In this way an 'effective Rayleigh number' Ra^* may be defined for deep water and a conjectured approximate instability criterion might be $Ra^* \gtrsim 10^3$.

At times sufficiently large that $La\,t > \frac{1}{4}$ we take $d = \kappa^{-1}$ in (4.1) to find that

$$Ra^* \approx a^2\omega u_*^2/\kappa^2\nu_T^3.$$

If we use the estimates quoted by Stewart (1967) for fully developed seas, that

$$a \sim 0{\cdot}2U_w^2/g, \quad \omega \sim g/U_w, \quad \kappa \sim g/U_w^2, \tag{4.3a}$$

which were employed by Leibovich & Radhakrishnan to estimate the eddy viscosity as

$$\nu_T \sim 2{\cdot}3\times10^{-5}U_w^3/g, \tag{4.3b}$$

and add their result

$$U_w \approx 660u_*, \tag{4.3c}$$

we recover $Ra^* \approx 7{\cdot}6\times10^6$, which far exceeds the conjectured critical value for instability. We note that for these estimates the Langmuir number La is $3{\cdot}6\times10^{-4}$.

For times such that $La\,t < \frac{1}{4}$, the expression (4.1) for Ra reduces to the remarkably simple formula $Ra^* = (4t)^2$. On reverting to dimensional time t_1 by using

$$t = t_1(a\kappa u_*)(\omega/\nu_T)^{\frac{1}{2}}$$

we obtain

$$Ra^* = 0{\cdot}064(gt_1/U_w)^2.$$

Accordingly, a notional critical value of 10^3 is exceeded after the time

$$t_{1c} \approx 1{\cdot}1\times10^2(U_w/g).$$

For a wind of 10 m s^{-1} 'switched on' at $t = 0$, the instability should therefore occur after a time of about two minutes. However, an odd feature of the result is that the onset time t_{1c} is even less for lighter winds! This anomaly is partly attributable to the assumed strong dependence on U_w of the estimate for ν_T; a *constant* viscosity ν_T would yield the opposite conclusion. Also, the properties of the wave field have been assumed independent of U_w. It must be remembered that t_{1c} is the time required to 'trigger' the instability: a further period of growth will be necessary before the disturbance becomes observable. This is considered in §5.

It is anticipated that the linearly most unstable cells are those aligned parallel to the wind, as assumed in the above analysis. A corresponding analysis for obliquely inclined cells will incorporate three additional features: a reduction of the effective Rayleigh number by $\cos^2\theta$, where θ is the angle of inclination of the cells to the wind direction; convection and shearing of the cellular structure by the component $\bar{u}\sin\theta$ perpendicular to the cells, and a term like $W\bar{u}_{zz}\sin\theta$ in the vorticity equation deriving from distortion of vorticity associated with the component $\bar{u}\sin\theta$. It is anticipated that the first two of these factors will tend to reduce the growth rate of the present instability; the last is connected with a Tollmien–Schlichting mode of instability in shear flows for which the most unstable waves are those with crests perpendicular to the wind direction – an altogether different phenomenon from that under discussion here.

Finally, the spacing of Langmuir cells may be estimated on the assumption that this corresponds to the wavelength $2\pi/l$ of the (instantaneously) most unstable linear disturbance. Again it seems plausible to choose the length scale κ^{-1} as an appropriate measure of effective depth (after sufficient time has elapsed for the notional boundary-layer thickness of the wind-driven current to exceed this value). In this case classical results for the Bénard problem (Chandrasekhar 1961, p. 43) predict a spanwise wavelength, comprising two counter-rotating cells, of about twice the length scale κ^{-1}.

At smaller times, the characteristic length scale is $2\kappa^{-1}(La\,t)^{\frac{1}{2}}$ rather than κ^{-1} and the first and smallest unstable spacing, corresponding to the critical time t_{1c}, is about $0{\cdot}2\kappa^{-1}$ for typical values of La. So the 'instantaneously most unstable' spacing increases with t from $0{\cdot}2\kappa^{-1}$ to $2\kappa^{-1}$ approximately.

5. The unsteady problem

The quasi-steady assumption for the mean flow ceases to be a satisfactory approximation when the predicted growth (or decay) rate of the disturbance is small and also at small times when $\bar{u}(z,t)$ varies particularly rapidly with t (see, for instance, Gresho & Sani 1971). For this reason and to indicate the possible course of future numerical work on the problem, attention is now focused on Foster's (1965, 1968) studies of the onset of convection in cooled fluid layers. This work closely parallels the initial-value/stability problem specified by (3.1) and (3.2) and it does not employ the quasi-steady assumption used above. A corresponding study of rotational Couette flow with an impulsively started inner cylinder has been made by Chen & Kirchner (1971) but their work need not be considered in detail here.

Foster (1965) deals with initially isothermal layers of finite depth subject to a temperature reduction imposed on the upper surface at $t = 0$. His governing equations are entirely analogous to (3.2) provided our $\mathscr{U}_z$ is taken to be constant ($= K$) over the entire fluid layer and provided his Prandtl number is unity. As above, the effective Rayleigh number Ra is $La^{-2}KH^4$. Foster examines two cases corresponding to the boundary conditions

$$(A)\ \ \bar{u}(0,t) = \begin{cases} 0 & (t < 0) \\ 1 & (t \geqslant 0) \end{cases}, \quad \bar{u}(-H,t) = 0 \quad (\text{all } t),$$

$$(B)\ \ \bar{u}(0,t) = \begin{cases} 0 & (t < 0) \\ t & (t \geqslant 0) \end{cases}, \quad \bar{u}(-H,t) = 0 \quad (\text{all } t),$$

and with free boundaries at $z = 0,\ -H$, where

$$\Psi = \partial^2\Psi/\partial z^2 = 0, \quad \hat{u} = 0.$$

The present boundary condition $\bar{u}_z = 1$ may be thought of as intermediate to these two cases, for it corresponds to $\bar{u}(0,t) \propto t^{\frac{1}{2}}$ [see (3.1)]. Also, we require $\partial\hat{u}/\partial z$ rather than $\hat{u}$ to vanish on the boundary $z = 0$, but Foster (1968) shows that this change does not greatly influence the results.

Foster (1968) considers a semi-infinite fluid subject to surface cooling as in cases A and B and with various boundary conditions corresponding to a rigid or free, conducting or insulating boundary. The present boundary conditions on $\hat{u}$ and Ψ correspond to a free insulating boundary.

In these unsteady problems, the time dependence is no longer exponential and solutions must be obtained numerically for specified initial disturbances. As a stability criterion, Foster employs the concept of a 'nominal critical time'. This is the time within which (a suitable average† of) a disturbance with prescribed spanwise wavenumber l is amplified by a chosen factor; and Foster considers factors ranging from

† Foster's average $\bar{w}(t)$ is here equivalent to $[I(t)/I(0)]^{\frac{1}{2}}$, where $I(t) = \int_0^\infty \Psi^2(z,t)\,dz$.

10 up to 10^8. In both his papers, Foster's computations demonstrate instability for all cases considered. Of course, for finite layers a critical Rayleigh number exists; but his computations mainly concern values of Ra considerably greater than this. For a fluid of infinite depth the Rayleigh number may be 'scaled out' of the equations, and instability *must* occur. However, in this latter respect our analogy with Foster's work fails; for the requirement that $\mathscr{U}_z$ remains constant is acceptable only for depths up to κ^{-1} at most, since $\mathscr{U}_z$ must approach zero for depths greater than this. Accordingly a critical 'effective Rayleigh number' will always exist in the present problem, as proposed in § 4.

Of particular interest are Foster's estimates of the most unstable wavenumber and the time required for the instability to become observable. This information may be deduced from Foster's (1968) paper. To conform with Foster's equations, we change to new dimensionless co-ordinates (z_2, t_2) of depth and time, defined as

$$z_2 = K^{\frac{1}{2}} La^{-\frac{1}{2}} z = K^{\frac{1}{2}} La^{-\frac{1}{2}} (\kappa z_1),$$
$$t_2 = Kt = K\,La^{-1} (\kappa^2 \nu_T t_1),$$

where (z_1, t_1) denote dimensional depth and time. The dimensionless variables $\hat{u}$ and $\overline{u}$ are also scaled by $(K\,La)^{\frac{1}{2}}$ and Ψ by $(i/l)(K\,La)^{\frac{1}{2}}$. As suggested above, the constant $\mathscr{U}_z = K$ may be taken as unity without loss of generality.

Foster's figures 3 and 5 show the minimum 'nominal critical time' t_{2c} for growth by factors of 10 and 10^8 and with various boundary conditions plotted against the Prandtl number $\mathscr{P}$ for cases A and B. For a growth factor of 10 at $\mathscr{P} = 1$, t_{2c} is about 13 for case B and 28 for case A; for a growth factor of 10^8 at $\mathscr{P} = 1$, t_{2c} is about 32 for case B and 160 for case A. These values correspond to a 'free and insulating' boundary as required by the present problem; but the values do not alter much for the other boundary conditions shown. Since the present boundary condition $\overline{u}(0,t) = t^{\frac{1}{2}}$ is intermediate between cases A and B, a reasonable conjecture for this case is $t_{2c} \approx 20$ for a growth factor of 10 and $t_{2c} \approx 100$ for a growth factor of 10^8.

For the values quoted in (4.2) as typical of wind speeds of 10 m s^{-1}, the corresponding dimensional times are 5·5 and 28 min respectively. (At those times, the respective thicknesses of the wind-driven boundary layer estimated by $2(La\,t)^{\frac{1}{2}}\kappa^{-1}$ are about $0{\cdot}6\kappa^{-1}$ and $1{\cdot}3\kappa^{-1}$.) That is to say, if a wind suddenly blows up (or changes direction?) it should take several minutes for the Langmuir circulations to become established. This is broadly in accord with observation and consistent with our estimate of 2 min for onset of instability using the quasi-steady approximation.

Foster's figures 4 and 6 show the critical horizontal wavenumber α_c (i.e. that which attains a prescribed growth factor of 10^8 in the shortest time) against Prandtl number. For $\mathscr{P} = 1$ and a free insulating surface, α_c is 0·4 for case B and 0·14 for case A. Supposing that the average $\alpha_c \approx 0{\cdot}27$ is appropriate for the present boundary condition on $\overline{u}$, we may estimate the dimensional critical wavelength as

$$\lambda_c = \frac{2\pi}{0{\cdot}27} La^{\frac{1}{2}} \kappa^{-1} = \frac{2\pi}{0{\cdot}27} \left(\frac{\nu_T}{\kappa a u_*}\right)^{\frac{1}{2}} \left(\frac{\nu_T}{\omega}\right)^{\frac{1}{2}}.$$

Incorporating dependence on wind speed as given by (4.3) the predicted spacing of Langmuir cells is $\lambda_c = 0{\cdot}45 U_w^2/g$: this is 4·6 m at wind speeds of 10 m s^{-1}.

The quasi-steady theory predicts a most unstable wavelength of $2\kappa^{-1}$ whenever the thickness of the wind-driven boundary layer grows beyond the depth scale κ^{-1}

of the pre-existing Stokes drift. That the estimate $0{\cdot}44\kappa^{-1}$ of unsteady theory (with the huge growth factor 10^8) is substantially smaller than this suggests that the spacing of Langmuir circulations may be determined while the wind-driven boundary layer is still quite thin compared with the Stokes-drift depth scale κ^{-1}. This lends some credence to our assumption that $\mathscr{U}_z$ may be treated as constant over depths of interest; but it must be admitted that this is a rather crude approximation.

Further numerical calculations directly appropriate to the present problem are desirable. It is a simple matter to calculate from (2.3) the actual Stokes drift $\mathscr{U}(z)$ corresponding to particular directional wave spectra for use in such computations (see appendix). Also, it would be worthwhile to consider the 'complete' initial-value problem in which also the wave spectrum evolves with time. For it is artificial to assume, as here, that the wave field may be prescribed independently of the wind: this is justifiable only if the waves attain a quasi-equilibrium state before the onset of circulations, but it is not clear that this is so.

6. Discussion

Although the mathematical problem is analogous to that for the onset of thermal convection, the physical processes causing the instability are quite different. The following explanation yields a clear understanding of the mechanism.

(i) Initially weak spanwise-periodic circulations give rise to variations $\hat{u}$ of the downwind velocity by advection of the developing mean Eulerian profile $\bar{u}(z,t)$. This is accomplished by the term $\Psi_y \bar{u}_z$ of (3.2).

(ii) The spanwise variation of $\hat{u}$ implies a periodic distribution of vertical vorticity: $\hat{u}_y$. But vorticity is convected by the Stokes drift $\mathscr{U}(z)$ since, in the absence of viscosity, material lines and vortex lines must coincide. Accordingly, the positive gradient $\mathscr{U}_z$ of the Stokes drift 'tilts' vertical vorticity to generate longitudinal (x) vorticity of a sense which reinforces the initial circulations postulated in (i). This tilting of vertical vorticity is represented by the term $\hat{u}_y \mathscr{U}_z$ of (3.2).

(iii) The diffusive and dissipative roles of (eddy) viscosity will tend to inhibit the inviscid instability mechanism of (i) and (ii), but La is usually sufficiently small for instability still to occur. The downwind velocity perturbation $\hat{u}$ according to inviscid theory is precisely zero at the free surface; but the viscous solutions display a structure in qualitative agreement with observed Langmuir circulations.

It is noteworthy that process (ii) is distinct from that identified by Leibovich & Ulrich as responsible for the initial growth of circulations forced by interacting wave pairs. Then the Stokes drift $\mathscr{U}$ is periodic in y and longitudinal vorticity is first generated by distortion of the mean spanwise vorticity $\bar{u}_z$ by this $\mathscr{U}$. The latter process corresponds to the term $\mathscr{U}_y u_z$ of (2.2), which is taken to be zero in the present work. Both processes are represented in the nonlinear solutions of Craik & Leibovich and Leibovich & Radhakrishnan for the wave-pair model.

Garrett's instability model also incorporates process (ii), albeit imprecisely and concealed within his averaging procedures. But process (i) is absent from his work since he regarded the downstream current $\bar{u}+\epsilon\hat{u}$ as independent of depth though varying in the spanwise direction. In fact, Garrett's mechanism is effectively independent of the wind, except in so far as it determines the directional properties of the wave field. This omission necessitated the introduction of distributed wave dissipation to complete the 'feedback cycle'. This is supported by the argument that, since wave amplitudes should increase in regions of surface convergence, dissipative processes must transfer more momentum from waves to downwind current in such regions. This would have the effect of reinforcing the variations $\hat{u}$ in downwind current.

While it is good to know that such variations in wave amplitude tend to support rather than suppress the instability, this effect is certain to be small compared with process (i). For, with currents of the magnitude assumed here, variations in wave amplitude would yield a contribution of higher order (i.e. smaller) than the terms retained in our governing equations (2.1). This is also true of the process suggested by Kraus (1967) involving damping of short waves by a surface slick located in zones of convergence.

Gammelsrød's theory is basically a special case of the instability discussed by Faller (1964, 1966) in which the depth scale is sufficiently small that the mean motion is a unidirectional and uniform shear flow. Thus Gammelsrød's analysis includes our process (i), for the case of constant mean shear $\bar{u}_z$, but not process (ii) for the waves play no part in his model. Instead, spanwise variations of longitudinal vorticity derive from the uniform vertical vorticity field of the earth's rotation. This vertical vorticity is distorted by the z derivative of the spanwise-periodic downstream velocity u so as to enhance the circulations; and these circulations in turn maintain $\hat{u}$ by process (i).

The strengths of process (ii) and Gammelsrød's Coriolis mechanism are readily compared. The rate of production of longitudinal vorticity by (ii) is proportional to $\hat{u}_y \mathscr{U}_z$ and that by the Coriolis mechanism is proportional to $2\hat{u}_z \Omega$, where Ω is the earth's angular velocity. If y and z derivatives are comparable in magnitude, as for typical Langmuir circulations, the relative strength of the processes is expressed by the ratio $(a\kappa)^2 \omega : 2\Omega$. For typical waves $a\kappa \approx 0{\cdot}2$ and $\omega \approx 1\ \mathrm{s}^{-1}$ while $\Omega = 1{\cdot}46 \times 10^{-4}\,\mathrm{s}^{-1}$, giving a ratio of about 150:1. The one situation in which Gammelsrød's mechanism may be significant is that of cells very widely spaced compared with the characteristic depth κ^{-1} since for these $\hat{u}_y$ may be small compared with $\hat{u}_z$ in the region of vorticity production. But Gammelsrød's treatment of viscous terms is imprecise and it is not clear that his inviscid mechanism is sufficiently strong to overcome viscous dissipation for such widely spaced cells. The application of his model to larger-scale atmospheric phenomena is perhaps more convincing; but the superficial comparison of 'cloud streets' and oceanic Langmuir circulations does not seem to withstand closer scrutiny.

On the other hand, it is quite possible that the modification of the wind-driven current by the Coriolis force (i.e. the establishment of an Ekman spiral) may somewhat alter the structure of Langmuir circulations at depths of order $(\nu/\Omega)^{\frac{1}{2}}$ or more. But the production of periodic longitudinal vorticity by processes (i) and (ii) is strongest near the surface, where the current is parallel to the wind direction; and these are likely to remain the dominant driving mechanisms of the circulations.

When the disturbances grow beyond the range of validity of the present linear theory, their form should resemble the parallel rolls of nonlinear thermal convection as studied by Elder (1968), Robinson (1976) and others, for the nonlinear equations (2.1) are analogous to their equations, just as in the linear case. Similarities should also exist with the nonlinear solutions of Liu & Chen (1973) for time-dependent rotational Couette flow. Further, although Leibovich & Radhakrishnan's (1977) solutions correspond to a Stokes drift with both a mean and a y-periodic part, the structure of their solutions will be similar to that of nonlinear disturbances with a y-independent Stokes drift. However, the concentration of x and z momentum in the vicinity of regions of surface convergence is likely to be somewhat diminished in the latter case. It is hoped that such solutions will be available in the near future.

There is some evidence to suggest that sufficiently strong circulations may exhibit 'secondary instability' yielding structures akin to penetrative thermal plumes, as in the experiments and computations of Elder (1968), Robinson (1976) and others. The pattern of windrows on the surface of a lake or ocean is frequently rather irregular and continuously evolving: it is a reasonable conjecture that this state may correspond to such unstable circulations. It also seems likely that the Craik–Leibovich model of interacting wave pairs will contribute significant 'forced solutions' which may fluctuate on a time scale long compared with the characteristic wave period; for the phases of the driving waves will vary randomly on this time scale. A qualitative picture thereby emerges of two possible regimes: one of regularly spaced cells and another, at greater wind speeds or later times, of quasi-turbulent fluctuations with strong spanwise variations in longitudinal vorticity.

A major factor which has yet to be investigated in detail is the influence of density stratification upon the Langmuir circulations and, conversely, the role of such circulations in establishing the mixed layer above the thermocline. It seems likely that a pre-existing stable stratification will inhibit the circulations at wind speeds below a certain critical value, but that above this critical wind speed the circulations may play a dominant part in creating the mixed layer and in establishing the position of the thermocline.

This work was stimulated by an interesting four-cornered correspondence with S. Leibovich, C. Garrett and R. T. Pollard.

Appendix. The Stokes drift

It has been assumed that the Stokes drift varies linearly with depth, which is of course only a rough approximation to reality (although probably no worse than the assumption of constant eddy viscosity!). We here consider the form of the Stokes drift for typical wave spectra. The usual dimensional Stokes drift, U_s say, is related to the dimensionless $\mathscr{U}(z)$ defined in (2.3) by $U_s = a^2\omega\kappa\mathscr{U}$, where a and ω are the characteristic wave amplitude and frequency used for non-dimensionalization and $\kappa = \omega^2/g$ is the characteristic wavenumber. The dimensional depth is $\hat{z} = \kappa^{-1}z$. With the estimates (4.3a) for fully developed seas this is just $U_s \approx 0{\cdot}04U_w\mathscr{U}(z)$, where U_w is the wind speed.

Kenyon (1969) has calculated $U_s(\hat{z})$ for various empirical spectra. In an integral form equivalent to (2.3) this is

$$U_s(\hat{z}) = \frac{2}{\rho g^2}\int_0^\infty f(\omega)\,\omega^3 \exp\left(-2\omega^2\hat{z}/g\right)d\omega,$$

where $f(\omega)$ is the one-dimensional frequency spectrum defined by

$$\rho g\overline{\zeta^2} = \int_0^\infty f(\omega)\,d\omega,$$

$\overline{\zeta^2}$ being the mean-square surface elevation.

For the case

$$f(\omega) = \alpha\rho g^3\omega^{-5}\exp[-\beta(g/U_w\omega)^2]$$

with the constants α and β chosen as 0·028 and 1·93 respectively to fit observational data (Pierson & Moskowitz 1964), Kenyon gives the solution

$$U_s(\hat{z}) = 3{\cdot}58\times10^{-2}U_w\exp[-3{\cdot}93(g\hat{z})^{\frac{1}{2}}/U_w].$$

The Stokes drift at the surface $\hat{z} = 0$ is here about 3·5 % of the wind speed and it decays rapidly with depth $\hat{z}$. Indeed, so sharp is the decay that the gradient of U_s is theoretically infinite at the surface. This property holds for *all* solutions corresponding to $f(\omega)\sim\omega^{-n}$ ($n \leqslant 6$) as $\omega\rightarrow\infty$, since the rapid variation is attributable to the short-wave contributions. In practice, however, the spectrum will be modified at large ω by the effects of capillarity and of viscosity, so that a short-wave cut-off frequency must exist. It is consistent with our assumed value of the eddy viscosity (4.3*b*) to impose a cut-off frequency at around 40 s^{-1}, corresponding to waves a few centimetres long (for which the 'wave Reynolds number' $\omega/\kappa^2\nu_T$ is around unity). This ensures that the gradient of the Stokes drift remains finite at the surface, but its value there is still likely to be rather large compared with that at depths of a few metres.

It is also worth noting that if the above spectrum $f(\omega)$ is modified to include a further factor $\exp[-2\omega^2z_0/g]$, where z_0 is some constant, the gradient of the Stokes drift varies much less rapidly near the surface and there is no need to invoke a cut-off frequency. In this case $dU_s/d\hat{z}$ varies nearly as $(z_0+z)^{-\frac{1}{2}}$ at depths small compared with U_w^2/g.

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